

Double-beta decay matrix elements for ^{76}Ge

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Abstract. Double-beta decay matrix elements (ME) for ^{76}Ge are calculated with different quasi-random phase approximation (QRPA)-based methods. First, the ME for the two-neutrino mode are computed using two choices for the single-particle (s.p.) basis: i) $2 - 4\hbar\omega$ full shells and ii) $3 - 4\hbar\omega$ full shells. When calculated with the renormalized QRPA (RQRPA) and full-RQRPA their values are rather dependent on the size of the single-particle basis used, while calculated with proton-neutron QRPA (pnQRPA) and second-QRPA approaches such a dependence was found to be small. The Ikeda sum rule was well fulfilled within pnQRPA for both choices of the s.p. basis and with a good approximation within second-QRPA, while the RQRPA and full-RQRPA methods give deviations up to 21%. Further, the ME for the neutrinoless mode are calculated with the pnQRPA, RQRPA and full-RQRPA methods. They all give close results for the calculation with the smaller basis (i), while for the larger basis (ii), the results differ significantly either from one method to another or within the same method. Finally, using the most recent experimental limit for the $0\nu\beta\beta$ decay half-life of ^{76}Ge a critical discussion on the upper limits for the neutrino mass parameter obtained with different theoretical approaches is given.

PACS. 21.60.Jz Hartree-Fock and random-phase approximations – 23.40.Hc Relation with nuclear matrix elements and nuclear structure – 23.40.Bw Weak interaction and lepton (including neutrino) aspects

1 Introduction

Since the nuclei which undergo a $\beta\beta$ -decay are generally rather far from the closed shells, the QRPA-based methods have been extensively employed for computing ME involved in the theoretical description of this process [1–27]. Moreover, in spite of the recent progress of the shell-model and/or Monte Carlo shell model techniques [28] these methods also remain, at least for the next future, the only available for treating nuclear systems which are far away from the closed shells. The pnQRPA [1] was the first adaptation of the standard QRPA for nuclear charge-changing processes. One of its most important achievements was after the pioneering work of [3] the success in explaining the suppression mechanism of the two-neutrino double-beta ($2\nu\beta\beta$) decay ME [4–7], reducing thus the large discrepancy existing until that moment between the theoretical and experimental $\beta\beta$ -decay half-lives. However, this method faces the problem of a strong dependence of these ME on the renormalization of the particle-particle component of the residual interaction. Namely, if one represents the ME as function of the particle-particle interaction strength (usually denoted by g_{pp}), one observes that

they decrease rapidly and cross through zero in a region of physical values of this constant, making the task of fixing it adequately difficult. To overcome this problem several further developments of this method have been advanced during the recent past. We remind here: the appropriate treatment of the particle number non-conservation [10, 13], the inclusion of the proton-neutron pairing [19], the double commutator method [12, 14], computation of the transitions to excited final states [14, 15, 24] as well as the development of approaches going beyond the quasi-boson approximation [9, 16, 18, 20, 27]. At this point it is worth mentioning a nice feature of these higher-order QRPA approaches: the like- and unlike-nucleon residual interactions appear both in the next higher-order terms beyond pnQRPA, obtaining thus a more realistic picture of the competition between them in producing a $\beta\beta$ -decay. As a result, calculated with these methods the ME become more stable against g_{pp} and the RPA breakdown point is shifted towards the region of unphysical values of this constant. This is why, the further improvement of such approaches seems to be the most promising line of development for treating the nuclear ME involved in the $\beta\beta$ -decay process.

The first method which has included higher-order terms beyond pnQRPA was developed in [9] and further, applied with some modifications in [15–17, 22]. In this ap-

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proach the extension of the pnQRPA was done using a boson expansion of both the phonon operators and transition β^\pm operators and retaining the next order in this expansion beyond the quasi-boson approximation (QBA). Also, this method allowed, for the first time, the computation of $\beta\beta$ -decay rates to excited final states. An alternative approach for extending pnQRPA is based on the idea of replacing the uncorrelated QRPA ground state (g.s.) by a correlated g.s., in the calculation of the expectation value of the commutator of the two bifermion operators involved in the derivation of the QRPA equations. The expectation values of the number operator in the QRPA correlated g.s. are introduced in the quasi-boson commutators of the pair operators and this leads to a renormalization of the QRPA forward- and backward-going amplitudes. This method (called RQRPA) was first developed in refs. [29–31] for the standard QRPA and adapted later on for charge-changing processes in ref. [18]. Within the RQRPA a stabilization of the ME against g_{pp} and a shift of the RPA breakdown point towards larger (unphysical) values of this constant are also observed. However, this method has a main inconvenience consisting in an undesirable violation of the Ikeda sum rule (ISR). Some refinements in the way of calculating the averages of the quasiparticle number operator are proposed [20–23], but they result in a rather small reduction of the violation.

In this paper we want to make a study of the $\beta\beta$ -decay nuclear ME of ^{76}Ge calculated with different QRPA-based methods with the same set of parameters and for both two neutrino and neutrinoless modes. The motivation of such a study is given by some discrepancies concerning their values which are still found in the literature, where similar calculations have been performed. First, we calculated the nuclear ME involved in the $2\nu\beta\beta$ -decay mode using the pnQRPA, RQRPA, full-RQRPA and second-QRPA methods. One of our goals was to see to what extent the size of the single-particle (s.p.) basis influences the values of these ME and how one can explain the differences between various calculations. A similar study has been made in [20] but only for the neutrinoless mode. Another point we have focused on was to check the Ikeda sum rule (ISR) in the framework of the above mentioned methods. Particularly, we would like to compare, under the same conditions of calculation (*i.e.* same parameters and s.p. basis), the various degrees of deviations obtained with these different methods and give possible explanations for the differences. Further, the ME for the neutrinoless mode are calculated with the pnQRPA, RQRPA and full-RQRPA methods. The results are found to be close to each other for all three methods in the case we used a smaller s.p. basis (9 levels), while for a larger one (12 levels) the results differ significantly either within the same method or from one method to another, for the two choices of the s.p. basis. Then, using the most recent experimental results for the two-neutrino and neutrinoless $\beta\beta$ -decay half-lives of ^{76}Ge [32,33], we fixed first the g_{pp} constant and then extracted new limits for the neutrino mass parameter. Finally, we give a critical discussion on the values of this parameter obtained with different theoretical methods. The paper

is organized as follows: in section 2 we will give a short comparative description of the QRPA-based methods that we used for the calculation. The results are presented in section 3 and section 4 is devoted to the conclusions.

2 Formalism

In the QRPA-based methods one assumes the nuclear motion to be harmonic and the excitation QRPA operator may have the following general expression:

$$\Gamma_{JM^\pi}^{m+} = \sum_{k,l,\mu\leq\mu'} \left[X_{\mu\mu'}^m(k,l,J^\pi) A_{\mu\mu'}^\dagger(k,l,J,M) + Y_{\mu\mu'}^m(k,l,J^\pi) \tilde{A}_{\mu\mu'}(k,l,J,M) \right]. \quad (2.1)$$

Here the summation is taken with $k \leq l$ if $\mu = \mu'$. X^m and Y^m are the forward- and backward-going QRPA amplitudes and A, A^\dagger the pair quasiparticle operators coupled to angular momentum J and projection M :

$$A_{\mu\mu'}^\dagger(k,l,J,M) = \mathcal{N}(k\mu, l\mu') \sum_{m_k, m_l} C_{j_k m_k j_l m_l}^{JM} a_{\mu k m_k}^\dagger a_{\mu' l m_l}^\dagger, \\ \tilde{A}_{\mu\mu'}(k,l,J,M) = (-)^{J-M} A_{\mu\mu'}(k,l,J,-M), \quad (2.2)$$

\mathcal{N} is a normalization constant, which is different from unity only in case when both quasiparticles are in the same shell [20], $\mu, \mu' = 1, 2$ and $1 \equiv$ protons, $2 \equiv$ neutrons. Using the equation of motion method one can derive the pnQRPA equations which, in the matrix representation, may be written as

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}_{J^\pi} \begin{pmatrix} X^m \\ Y^m \end{pmatrix} = \Omega_{J^\pi}^m \begin{pmatrix} \mathcal{U} & 0 \\ 0 & -\mathcal{U} \end{pmatrix}_{J^\pi} \begin{pmatrix} X^m \\ Y^m \end{pmatrix}, \quad (2.3)$$

where the matrices \mathcal{A}, \mathcal{B} and \mathcal{U} have the following expressions:

$$\mathcal{A}_J(\mu k, \nu l; \mu' k', \nu' l') = \\ \langle 0_{\text{RPA}}^+ | \left[A_{\mu\nu}(k,l,J,M), [\hat{H}, A_{\mu'\nu'}^\dagger(k',l',J,M)] \right] | 0_{\text{RPA}}^+ \rangle, \\ \mathcal{B}_J(\mu k, \nu l; \mu' k'; \nu' l') = \\ \langle 0_{\text{RPA}}^+ | \left[A_{\mu\nu}(k,l,J,M), [\tilde{A}_{\mu'\nu'}(k',l',J,M), \hat{H}] \right] | 0_{\text{RPA}}^+ \rangle, \quad (2.4)$$

$$\mathcal{U} = \langle 0_{\text{RPA}}^+ | \left[A_{\mu\nu}(k,l,J,M), [A_{\mu'\nu'}^\dagger(k',l',J,M)] \right] | 0_{\text{RPA}}^+ \rangle. \quad (2.5)$$

Here the $\Omega_{J^\pi}^m$ are the QRPA excitation energies for the mode J^π .

Within the pnQRPA the QBA is assumed, *i.e.* the quasiparticle operators A, A^\dagger are bosons and satisfy exactly the boson commutation relations:

$$\left[A_{\mu\nu}(k,l,J,M), A_{\mu'\nu'}^\dagger(k',l',J,M) \right] = \mathcal{N}(k\mu, l\nu) \mathcal{N}(k'\mu', l'\nu') \\ \times (\delta_{\mu\mu'} \delta_{\nu\nu'} \delta_{kk'} \delta_{ll'} - \delta_{\mu\nu'} \delta_{\nu\mu'} \delta_{lk'} \delta_{kl'}) (-)^{j_k + j_l - J}. \quad (2.6)$$

In this way the Pauli principle is violated and this is a serious drawback of this method. To improve the situation in

the RQRPA method the A , A^\dagger operators are renormalized [18,20]:

$$\bar{A}_{\mu\mu'}(k, l, J, M) = D_{\mu k \nu k'}^{-1/2} A_{\mu\mu'}(k, l, J, M), \quad (2.7)$$

where the $D_{\mu k \nu k'}$ matrices are defined as follows:

$$\begin{aligned} D_{\mu k \nu k'} &= \mathcal{N}(k\mu, l\nu)\mathcal{N}(k'\mu', l'\nu') \\ &\times (\delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{kk'}\delta_{ll'} - \delta_{\mu\nu'}\delta_{\nu\mu'}\delta_{lk'}\delta_{kl'})(-)^{j_k+j_l-j} \\ &\times \left[1 - j_l^{-1}\langle 0_{\text{RPA}}^+ | [a_{\nu l}^\dagger a_{\nu l'}]_{00} | 0_{\text{RPA}}^+ \rangle \right. \\ &\quad \left. - j_k^{-1}\langle 0_{\text{RPA}}^+ | [a_{\mu k}^\dagger a_{\mu k'}]_{00} | 0_{\text{RPA}}^+ \rangle \right]. \end{aligned} \quad (2.8)$$

By inspecting (2.6) and (2.8) one observes that by this renormalization one goes beyond the QBA by taking into account the next terms in the commutator relations of the A , A^\dagger operators which are just, essentially, the proton and neutron number operators. It is worth mentioning that they are taken into account within RQRPA only by their averages on the RPA g.s.. The renormalization of the A , A^\dagger operators is further carried onto the RPA amplitudes, on the \mathcal{A} , \mathcal{B} matrices and on the RPA phonon operator also obtaining a renormalization of them:

$$\begin{aligned} \bar{X}^m &= D^{1/2} X^m; & \bar{Y}^m &= D^{1/2} Y^m; \\ \bar{A}^m &= D^{-1/2} A D^{-1/2}; & \bar{B}^m &= D^{-1/2} B D^{-1/2}, \end{aligned} \quad (2.9)$$

$$\begin{aligned} \Gamma_{JM}^{m+} &= \sum_{k,l,\mu\leq\mu'} \left[\bar{X}_{\mu\mu'}^m(k, l, J^\pi) \bar{A}_{\mu\mu'}^\dagger(k, l, J, M) \right. \\ &\quad \left. + \bar{Y}_{\mu\mu'}^m(k, l, J^\pi) \bar{A}_{\mu\mu'}(k, l, J, M) \right]. \end{aligned} \quad (2.10)$$

To calculate \bar{A} and \bar{B} , we need to determine the renormalization matrices D . This is done by solving a system of non-linear equations for them by an iterative numerical procedure. As input values one can use their expressions in which the averages of the number operators are replaced by the back-forwarded amplitudes obtained as initial solutions of the QRPA equation.

In QRPA-type methods, before starting the RPA procedure, we need the occupation amplitudes (u , v) and the quasiparticle energies, in order to get the image of the RPA operators in the quasiparticle representation. This is done by solving the HFB equations, which may include, in the general case, both like- and unlike-nucleon pairing. When one includes only like-nucleon pairing in these equations, the QRPA procedure described above was called RQRPA [18,20,25,27], while when both types of the pairing interaction are included it was called full-RQRPA [20,25]. On the other hand, if one takes the $D = 1$, we get back the QBA and these methods become pnQRPA and full-QRPA, respectively.

In the second-QRPA method the principle of including higher-order corrections to the pnQRPA and restoring partially the Pauli principle is different. Here, the two quasiparticle and the quasiparticle density dipole operators are expanded in a Beliaev-Zelevinski series [34]:

$$A_{1\mu}^\dagger(pn) = \sum_k \left(A_{k_1}^{(1,0)} \Gamma_{1\mu}^+(k) + A_{k_1}^{(0,1)} \tilde{T}_{1\mu}^+(k) \right), \quad (2.11)$$

$$\begin{aligned} B_{1\mu}^\dagger(pn) &= \sum_{k_1 k_2} \left(B_{k_1 k_2}^{(2,0)}(pn) [\Gamma_1^\dagger(k_1) \Gamma_2^\dagger(k_2)]_{1\mu} \right. \\ &\quad \left. + B_{k_1 k_2}^{(0,2)}(pn) [\Gamma_1(k_1) \Gamma_2(k_2)]_{1\mu} \right), \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} B_{1\mu}^\dagger(pn) &= \sum_{m_k, m_l} C_{j_p m_p j_n m_n}^{JM} a_{j_p m_p}^\dagger a_{j_n m_n}, \\ \tilde{B}_{1\mu}(pn) &= (-)^{J-M} B_{1\mu}(pn). \end{aligned} \quad (2.13)$$

The boson expansion coefficients $A^{(1,0)}$, $A^{(0,1)}$, $B^{(2,0)}$, $B^{(0,2)}$ are determined so that eqs. (2.11)-(2.12) are also valid for the corresponding ME in the boson basis.

Further, the transition β^\pm operators in the quasiparticle representation can be expressed in terms of the dipole operators $A_{1\mu}$ and $B_{1\mu}$:

$$\begin{aligned} \beta_\mu^-(k) &= \theta_k A_{1\mu}^\dagger(k) + \bar{\theta}_k \tilde{A}_{1\mu} + \eta_k B_{1\mu}^\dagger(k) + \bar{\eta}_k \tilde{B}_{1\mu}, \\ \beta_\mu^+(k) &= -\left(\bar{\theta}_k A_{1\mu}^\dagger(k) + \theta_k \tilde{A}_{1\mu} + \bar{\eta}_k B_{1\mu}^\dagger(k) + \eta_k \tilde{B}_{1\mu} \right), \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} \theta_k &= \frac{\hat{j}_p}{\sqrt{3}} \langle j_p || \sigma || j_n \rangle U_p V_n; \\ \bar{\theta}_k &= \frac{\hat{j}_p}{\sqrt{3}} \langle j_p || \sigma || j_n \rangle U_n V_p; \quad \hat{j} = \sqrt{2j+1}, \\ \eta_k &= \frac{\hat{j}_p}{\sqrt{3}} \langle j_p || \sigma || j_n \rangle U_p U_n; \quad \bar{\eta}_k = \frac{\hat{j}_p}{\sqrt{3}} \langle j_p || \sigma || j_n \rangle V_p V_n. \end{aligned} \quad (2.15)$$

Using the boson expansions (2.11)-(2.12), one also gets expressions of the transition operators beyond the quasiboson approximation. Thus, in the second-QRPA method, the higher-order corrections to the pnQRPA are introduced not only in the RPA wave functions (by improving the phonon operator with additional correlations), but also in the expressions of the β^\pm operators, and the procedure is now more consistent. The additional terms will have, of course, an influence on the ME calculation of these operators.

Further, we give the factorized forms of the two-neutrino and neutrinoless $\beta\beta$ -decay half-lives that we used in our calculations:

$$\left[T_{1/2}^{2\nu} \right]^{-1} = F^{2\nu} |M_{\text{GT}}^{2\nu}|^2, \quad (2.16)$$

where $F^{2\nu}$ is the lepton space phase and

$$M_{\text{GT}}^{2\nu} = \sum_{l,k} \frac{\langle 0_f^+ || \sigma \tau^- || 1+k \rangle \langle 1_k^+ | 1_l^+ \rangle \langle 1_l^+ || \sigma \tau^- || 0_i^+ \rangle}{E_l + Q_{\beta\beta}/2 + m_e - E_0}. \quad (2.17)$$

Table 1. Values of the renormalizing constants.

	No. levels	g_{ph}	g_{ph}^2	g_{pp}^*	g_{pair}^n	g_{pair}^p	g_{pair}^{pn}
^{76}Ge	9	1.0	0.8	1.0	1.230	1.057	2.071
	12	1.0	0.8	1.0	1.011	1.034	1.751
^{76}Se	9	1.0	0.8	1.0	1.138	1.214	1.653
	12	1.0	0.8	1.0	1.031	1.194	1.502

In (2.17) l, k denote the two different sets of 1^+ states in the odd-odd nucleus obtained with two separate RPA procedures applied onto the g.s. of the initial and final nuclei participating in the $\beta\beta$ -decay. E_l is energy of the l -th intermediate 1^+ state, and E_0 is the initial g.s. energy.

$$\left[T_{1/2}^{0\nu}\right]^{-1} = C_{mm} \left(\frac{\langle m_\nu \rangle}{m_e}\right)^2, \quad (2.18)$$

where $\langle m_\nu \rangle$ is the effective neutrino mass and

$$C_{mm} = F_1^{0\nu} \left(M_{\text{GT}}^{0\nu} - \left(\frac{g_\nu}{g_A}\right)^2 M_F^{0\nu} \right)^2 = F_1^{0\nu} \cdot (M^{0\nu})^2, \quad (2.19)$$

$F_1^{0\nu}$ is the phase-space integral and $M_{\text{GT}}^{0\nu}$ and $M_F^{0\nu}$ are Gamow-Teller and Fermi matrix elements.

3 Results

3.1 Two-neutrino double beta decay

First, we have performed calculations of the nuclear ME involved in the $2\nu\beta\beta$ decay mode of ^{76}Ge using the pnQRPA, RQRPA, full-RQRPA and second-QRPA methods. For the s.p. basis we used two choices. We included: i) the 12 levels belonging to the full sd, pf and sdg shells, taking thus ^{16}O as core and ii) the 9 levels belonging to the full pf and sdg shells and taking thus ^{40}Ca as core. The s.p. energies have been obtained by solving the Schrödinger equation with a Coulomb-corrected Woods-Saxon potential. For the residual two-body interaction there was taken the Brueckner G -matrix calculated from a Bonn-OBEP. The quasiparticle energies and the BCS occupation amplitudes were derived by solving the HFB equation without and with proton-neutron pairing, separately for the initial and final nuclei, with both choices of the s.p. basis. For a complete calculation we included in the model space the states with all the multiplicities J^π . The renormalization constants were chosen as follows: $g_{\text{pp}} = 1.0$ for all the multiplicities, except the 1^+ channel for which it was left as a free parameter, and $g_{\text{ph}} = 1.0$ for all the multiplicities except the 2^+ channel where it was fixed to 0.8, since for larger values the p-h interaction in this channel is too strong producing the collapse of the RPA procedure. The value of all the constants, including those which renormalize the pairing interactions are presented in the table 1.

In fig. 1 we displayed the $M_{\text{GT}}^{2\nu}$ (in MeV^{-1}) as function of g_{pp} calculated with pnQRPA and second-QRPA

methods. The two curves for each method represent the calculations performed with the two different s.p. basis. In the figure is also drawn the line representing the ME value corresponding to the latest experimental $2\nu\beta\beta$ -decay half-life of the ^{76}Ge , obtained by the Heidelberg-Moscow experiment: $T_{1/2}^{2\nu} = 1.55 \times 10^{21}$ y [33].

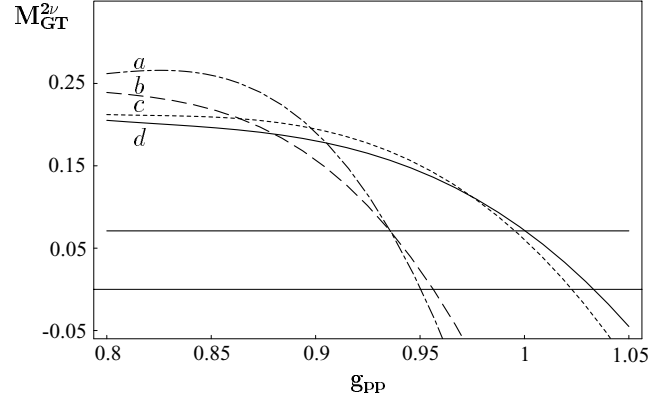


Fig. 1. a) pnQRPA (12); b) pnQRPA (9); c) second-QRPA (9); d) second-QRPA (12).

As was already observed in previous calculations [9, 15] the point where QRPA breaks down is pushed to higher values of g_{pp} in the framework of the second-QRPA method as compared with the pnQRPA. The calculation also shows that, within these two methods the values of the ME do not depend significantly on the size of the s.p. space, especially in the region around the experimental value. The values of g_{pp} which fit the best this experimental value are: 0.94 for both calculations performed with pnQRPA and 0.99 and 1.01 for the calculation with 12 and 9 levels, respectively performed with second-QRPA.

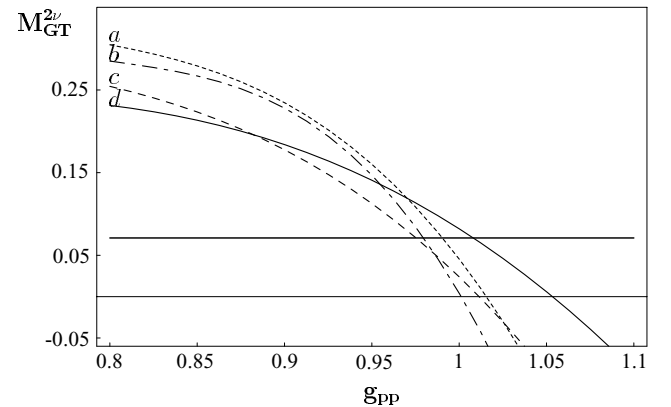


Fig. 2. a) full-RQRPA(12); b) full-RQRPA(9); c) RQRPA(9); d) RQRPA(12).

In fig. 2 the same ME but calculated with RQRPA and full-RQRPA methods are displayed. Contrary to the previous calculations, in this case the difference between the results obtained with different choices of the s.p. space,

Table 2. The numbers represent the deviations (in percents) from the ISR calculated within the specified methods. The first (second) numbers in the rows represent the calculation with a s.p. basis containing 12 (9) levels, respectively.

	pnQRPA		RQRPA		full-RQRPA		second-QRPA	
^{76}Ge	0.23	0.26	20.06	21.34	17.68	17.29	2.7	3.7
^{76}Se	0.41	0.48	19.66	19.94	17.12	16.91	3.13	4.18

within the same method, is rather large. Indeed at values of g_{pp} , where $M_{GT}^{2\nu}$ crosses the line representing the experimental result, the values for the ME, obtained with the same method, differ from each other by up to 40 – 50% when the two different choices of the s.p. basis are used. The values of g_{pp} for the best fits with the experimental value for the ME are: 0.977; 0.982 in the case of the RQRPA and 0.975; 1.012 in the case of the full-RQRPA for calculations including 9 and 12 levels in the s.p. basis, respectively. The differences persist when the s.p. basis was enlarged to 21 states. This different behavior of the calculation, obtained with RQRPA and full-RQRPA on one side, and with pnQRPA and second-QRPA on the other side, in connection to the choices of the s.p. basis, reflects the sensitivity of the former methods in computing the $M_{GT}^{2\nu}$ ME. One possible source of this sensitivity might have its origin in the numerical computation. Indeed, the self-consistent iteration procedures for solving the full RQRPA equations, for all the multiplicities, are very time-consuming and rather slow converging and might affect the precision of the calculation. For more reliable calculations improved numerical techniques are in our opinion further required.

On the other hand, there are some theoretical arguments which could explain the different results for the ME obtained with RQRPA-like methods as compared to the other two. We will discuss them later, after having discussed the ISR.

The ISR

$$S_- - S_+ = \sum_m |\langle 0_{g.s.}^+ || \beta_m^- || 1_m^+ \rangle|^2 - \sum_m |\langle 0_{g.s.}^+ || \beta_m^+ || 1_m^+ \rangle|^2 \quad (3.1)$$

was checked out in the framework of the four methods. The results are presented in table 2, where the percentages of deviation from the correct value for each method and choice of the basis are given. The first values in the row represent the calculations with a s.p. basis with 12 levels, while the second numbers refer to the same calculation, but with 9 levels.

One can see, as expected, that within the pnQRPA the ISR is very well fulfilled, while within RQRPA and full-RQRPA the deviations are between 17–21%. One can also observe that within second-QRPA the deviations from the ISR are rather small, confirming the result reported earlier in refs. [15,16], but at that time calculated including only the 1^+ channel and 9 levels in the s.p. basis. We should mention that in the present second-QRPA calculation we did not take into account the three boson states which may introduce undesirable spurious states in the QRPA

space. Including such states one also gets deviations up to 17% from the ISR. Looking for some theoretical arguments for a possible explanation of the different extent to which the ISR is fulfilled within the RQRPA and second-QRPA methods, one finds that one reason could be the existence of some inconsistencies of the RQRPA related to the way of partial restoration of the Pauli principle.

Indeed, as we already mentioned in section 2, within the RQRPA method the Pauli principle is partially restored for the operators A , A^\dagger , by taking into account the averages of the quasiparticle-number operators in their commutator relations. However, there is no justification to neglect them in the B , B^\dagger operator commutation relations. Within the second-QRPA higher-order corrections beyond pnQRPA are taken into account in the expression of these operators and moreover, such corrections are also introduced in the expressions of the β^\pm operators. The effect of the additional terms combined with a larger boson space (in the second-QRPA the boson space enlarges from one to two boson states) reflects in a positive contribution to the ISR. On the other hand it is known that RQRPA underestimates the ISR, so this could be one possible explanation why the ISR is better fulfilled within second-QRPA.

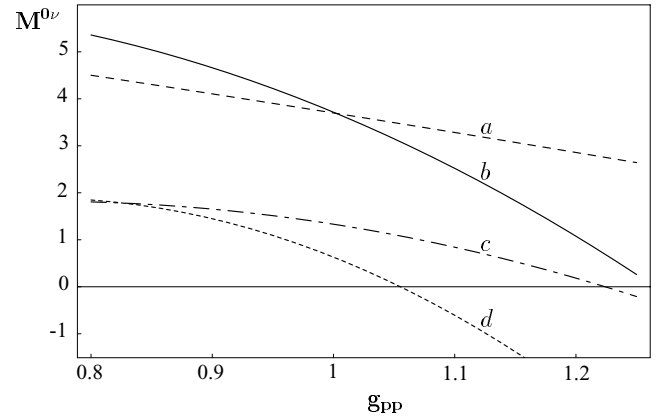


Fig. 3. a) RQRPA(9); b) pnQRPA(9); c) RQRPA(12); d) pnQRPA(12).

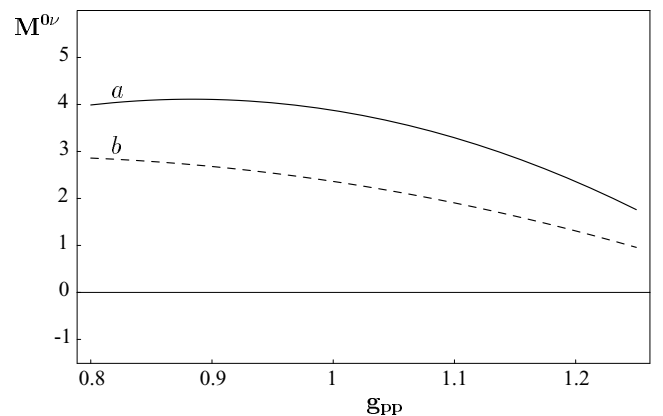


Fig. 4. a) full-RQRPA(9); b) full-RQRPA(12).

Table 3. Neutrinoless ME and upper limits for the neutrino mass parameter for ^{76}Ge , calculated with a phase space $F_1^{0\nu} = 6.31 \times 10^{-15} \text{ y}^{-1}$ [26] and $T_{1/2}^{0\nu} > 1.9 \times 10^{25} \text{ y}$ (90% c.l.), and $3.1 \cdot 10^{25} \text{ y}$ (68% c.l.) [33]. The non-dimensional values of various ME are taken and reconverted, when necessary, from the indicated references. For the present work two values, representing the calculation with 12 and 9 levels for the s.p. basis, are displayed.

	[8]	[11]	[2]	[28]	[25]	[35]	Present work	
$M^{0\nu}$	4.25	4.26	4.85	1.57	1.92	2.80	2.36(12)	3.62(9)
$\langle m_\nu \rangle$ (eV) 90%	0.345	0.328	0.304	0.940	0.768	0.527	0.625	0.407
$\langle m_\nu \rangle$ (eV) 68%	0.27	0.26	0.24	0.73	0.60	0.41	0.490	0.320

Another possible shortcoming of the RQRPA method is a lack of consistency between BCS and QRPA levels. While in the BCS still one assumes the g.s. to be the quasiparticle vacuum at the level of RQRPA we are dealing with the non-vanishing quasiparticle content of the g.s. due to the additional scattering terms taken into account in the commutation relations [27].

3.2 Neutrinoless double-beta decay

Further, we have performed a calculation of the neutrinoless ME using pnQRPA, RQRPA and full-RQRPA methods, also for the two s.p. basis. The $M^{0\nu}$ as function of g_{pp} calculated with the pnQRPA and RQRPA are displayed in fig. 3, while the same ME but calculated with the full-RQRPA are displayed in fig. 4. One observes that all the three methods used for calculation give different values of the ME for different choices of the s.p. basis. The differences between ME values calculated with 9 and 12 levels included in the s.p. basis, within the pnQRPA and RQRPA methods, are given by factors of 3 and 2.5, respectively, while for the full-RQRPA method the difference between the two calculations reduces to a factor of about 1.6. One also observes, that the values of the ME obtained with the three methods are close to each other (3.9-4.1) in the calculation with the smaller basis. When enlarging the basis to 21 levels, the result is close to that obtained with the basis with 12 levels. This again reveals the sensitivity of the RQRPA-type methods to the choice of the s.p. basis and seems to indicate a possible stabilization of the results for larger basis. However, a general conclusion about which basis is better to choose is difficult to give until we have not the whole image of a similar study performed on several other double-beta emitters. Another still open question is what are the results when a similar calculation is performed with the second-QRPA.

Finally, using the value of the g_{pp} constant, fixed from the $M_{GT}^{2\nu}$ calculation to fit the most recent half-life, *i.e.* 1.0 (very close to the average value between the two second-QRPA and full-RQRPA calculations), and using the most recent experimental limit of the neutrinoless mode half-life for the ^{76}Ge case (*i.e.* $> 1.9 \times 10^{25} \text{ y}$ (90% c.l.) reported by the Heidelberg-Moscow experiment [33]) we extract new upper limits for the neutrino mass parameter within the full-RQRPA method. We obtain the values = 0.407 eV and 0.625 eV, if we use in the calculation a s.p. basis with 9 and 12 levels, respectively. In addition, in table 3, besides our values for the ME and neutrino mass parameter,

we present results of other calculations found in the literature. For a direct comparison between various results we use non-dimensional values for all the ME taken from the references indicated in the table. Further, using the same phase space factor $F_1^{0\nu} = 6.31 \times 10^{-15} \text{ y}^{-1}$ [26] and the same half-life reported in ref. [33], we extracted upper limits for $\langle m_\nu \rangle$ corresponding to all values of the ME, in order to have a more complete image of the evolution of the theoretical predictions. Performed with pnQRPA, the older results are very similar, although they were calculated by different groups and with different numerical codes and parameters [8,11]. One also observes that their values are larger by a factor of about two than the values obtained by using the recent extensions of pnQRPA, RQRPA and full-RQRPA. However, it should be kept in mind that these last approaches do not fulfill the ISR and thus lead *a priori* to too small ME, *i.e.* too large neutrino mass limits. Our calculations performed with the larger s.p. basis give ME rather close to those of ref. [35] were corrections due to the nucleon currents such as weak magnetism and pseudoscalar couplings to the amplitude of $0\nu\beta\beta$ have been taken into account. The use of the smaller s.p. basis yields a value for the ME which is close to the earlier approaches [5,8,11]. It should be pointed out that corrections due to nucleonic currents mentioned above were not made in the present study.

On the other side there are the values of the ME calculated with the shell-model in refs. [2,28] which differ from each other by a factor of 3. However, in our opinion, these calculations performed with the shell model are not, at present, reliable enough. This has been stressed also by [36]. In ref. [2] the calculations were performed with a rather crude shell-model code in a weak coupling approximation, at the computer performances of that time. Also, in calculations of ref. [28] some important orbits are missing, like some spin orbit partners, resulting in a violation of the Ikeda sum rule of about 50%, which means it should be expected that they give too small ME.

4 Conclusions

We have performed a calculation of the two- and zero-neutrino $\beta\beta$ -decay matrix elements for the case of ^{76}Ge with the pnQRPA, second-QRPA, RQRPA and full-RQRPA methods, using two different choices of the s.p. basis. We can summarize the main results as follows.

i) For the $M_{GT}^{2\nu}$ we got a significant dependence of the results on the size of the s.p. basis, in the case of the

RQRPA and full-RQRPA methods, while the results obtained with pnQRPA and second-QRPA do not display such a dependence.

ii) For the neutrinoless decay mode all the three methods used for the calculation, *i.e.* pnQRPA, RQRPA and full-RQRPA, give differences between 9 and 12 level calculations by factors 1.6–3. The values of the ME obtained with the three methods are close to each other for the calculation with the smaller basis, while they differ significantly when the calculation is done with the larger basis. i) and ii) reveal a sensitivity of the RQRPA methods to the size of the s.p. basis which is used. This could have its root in the numerical double-iteration procedure used in RQRPA-type calculations and in our opinion further improvements should be done in this respect.

iii) We also checked the ISR within the four methods and found it to be fulfilled with a good approximation within the second-QRPA method, while with RQRPA and full-RQRPA the deviations are up to 21%. We found that this result is not much dependent on the size of the s.p. basis used. This result, besides the numerical arguments mentioned above, might also be explained by theoretical arguments related to the way the partial restoration of the Pauli principle is done within RQRPA. The restoration is made in the commutator relations of the operators A , A^\dagger by taking into account the averages of the quasiparticle-number operators in their commutator relations. However, there is no justification to neglect them in the B , B^\dagger operator commutation relations. However, this is done within the second-QRPA and moreover, in this method the next higher-order corrections beyond pnQRPA are also taken into account for the β^\pm operators. The additional terms give a positive contribution to the ISR, while as is known RQRPA underestimates the ISR.

iv) Using the most recent reported neutrinoless half-life limit, and using the value of g_{pp} fixed for the 2ν neutrino mode calculation, we extracted new upper limits for the neutrino mass parameter. A critical comparison between various values of the ME found in the literature was performed. One may conclude that the values of the ME involved in $0\nu\beta\beta$ -decay of ^{76}Ge can be reliably predicted within a factor of two.

Finally we would like to stress that considering the various approximations made in the different calculations (violation of Ikeda sum rule), and neglect of weak magnetism and pseudoscalar coupling in all approaches except in [35] (another 30%), the tendency goes to a variation within the different approaches of only a factor of 1.5, and to favouring the smaller deduced neutrino mass values.

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